Int J Theor Phys (2009) 48: 621–629
DOI 10.1007/s10773-008-9835-6

The Gravitational Waves from Test Particles around Black Holes Immersed in a Strong Magnetic Field

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Received: 27 April 2008 / Accepted: 19 August 2008 / Published online: 3 September 2008

Abstract In this paper, we calculate the gravitational waveform from free test particles around Schwarzschild black holes immersed in a uniform strong magnetic field. By comparing with the cases of the Schwarzschild black holes, we find that for stable circle orbits, magnetic field can amplify amplitude and frequency of gravitational waves (here after GWs). For other general orbits, the uniform magnetic field also can amplify amplitude of GWs, enhance energy radiation of GWs and make it to shift to higher frequency. Another obvious influence of magnetic field B is that it can change the form of $h_\times$ component of GWs.

Keywords Black hole physics · Gravitational waves · Magnetic fields

1 Introduction

The Einstein's theory of general relativity predicts the existence of gravitational waves. A very famous indirect evidence of the existence of GWs is the work of the binary pulsar PSR 1913+16 by Taylor, Wolszczan, Damour and Weisberg in 1992 [1]. But until today, people have still not detected GWs directly. So detecting of GWs and proving Einstein's general relativity are one of the most important science target in this century. Furthermore, GW can be a very important tool used in physics and astrophysics research. For this, some countries are launching several grant detecting projects, such as LISA (Laser Interferometer Space Antenna), LIGO (Laser Interferometer Gravitational Wave Observatory) and ASTROD (Astrodynamical Space Test of Relativity using Optical Devices) et al. to search GWs.

Because the compact binary systems are very good source of GWs, many people are interested in studying their dynamics and waveform at the post-Newtonian approximation [2–10]. At present the post-Newtonian (PN) approximation of binaries is only accurate to 3.5PN

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order, but the process of binaries merging is very relativistic, so that the post-Newtonian approximation perhaps is not enough accurate. Recently, some authors researched the match of the post-Newtonian approximation with high precision numerical relativity simulation and comparing their gravitational waveform [11–13].

On the other hand, Buonanno and Damour introduced a nonperturbative resummation of the two-body conservative dynamics, named effective-one-body (EOB) approach [14]. And then some works used this method to study the GWs forms of binaries were published lastly [15, 16].

Both the post-Newtonian Lagrangian formulation and the EOB approach [26, 27], they are not full in general relativity frame. But, if we assume mass of one star of the binary is small enough (cannot influence the massive star's motion and self-gravity can be ignored), then the two-body problem transform to one-body problem (test particle moving in known space-time), so that the model becomes simpler and can be solved in full relativistic frame.

Although the model of test particles around massive central bodies is very ideal, it still can present some important qualitative properties about realistic astrophysics objects. Some papers dealing with this field can be found in the last decade. For example, the GWs emitted from a spinning test particle have been calculated by several authors. Mino et al. discussed the GWs and energy emission produced by a spinning test particle falling into a Kerr black hoke along the symmetric axis of the space-time or moving circularly around it [17]. And Saijo et al. also calculated the GWs emitted from a spinning test particle plunding into a Kerr black hole from infinity [18]. Suziki and Maeda studied influence of chaotic orbits to the GWs' signature of spinning test particles in Schwarzschild space-time [19], and then these results were extended to Kerr space-time [20].

But all the above referred studies, didn't include the case of existence of magnetic field, and almost all realistic astrophysical black holes or neutron stars are magnetic. In general relativity, magnetic field also can change gravitational field, and influence the dynamics of objects in it. Accordingly, the existence of magnetic field would effect the GWs' signature.

In this paper, for the purpose of preliminary research about the effect of the magnetic field on the GWs, we consider a simple model of test particles around a Schwarzschild black hole immersed in an uniform strong magnetic field. By comparing with nonmagnetic cases, we analyze how the uniform magnetic field works on the gravitational waveform. Some distinct results are found, such as the magnetic fields can amplify amplitude and frequency of gravitational waves, change the shape of the $h_\times$ component of GWs obviously, make the energy radiation more larger and shift the energy radiation to higher frequency. Especially, for circle orbits, we analytically write out the expression of gravitational waveforms, and can clearly see how magnetic fields affect on the GWs.

Throughout the work we use geometric units $c = G = 1$, and take the signature of a metric as $(-,+,+,+)$. Greek subscripts run from 0 to 3, and Latin indexes run from 1 to 3.

2 Metric and Equations of Motions

It is well known that supermassive black holes in centers of galaxies are immersed in a strong magnetic field, so a model with a magnetic field has realistic astrophysics situations. The Ernst metric in four dimensions of a black hole in a uniform magnetic field can be written as follows [21],

$$ds^2 = \Lambda^2\left[\left(-1+\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\theta^2\right] + \frac{r^2\sin^2\theta}{\Lambda^2}d\phi^2, \qquad (1)$$

where the external magnetic field is determined by the parameter B,

$$\Lambda = 1 + B^2 r^2 \sin^2\theta. \tag{2}$$

Because the magnetic field is assumed exist everywhere in space, the above metric is not asymptotically flat. In present paper, we set $M = 1$, and even very strong magnetic field in centers of galaxies, corresponds to $B \ll M = 1$.

Now, we describe the motion equation of a test particle around this black hole. In general relativity, the dynamical equation of particles in space-time be given as follows,

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}. \tag{3}$$

Considering the symmetry of the metric, there are two integration constants:

$$E = -g_{tt}\dot{t}, \tag{4}$$

$$L = g_{\phi\phi}\dot{\phi}, \tag{5}$$

where E and L represent energy and z angular momentum of the system respectively. The system admits a third conservation, the so called four-velocity conservation,

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -1. \tag{6}$$

Because the GWs propagation isn't local, the time variable we used cannot be proper time τ, but coordinate time t. We must rewrite (3) as follows,

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{00} - 2\Gamma^i_{0j}\frac{dx^j}{dt} - \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} + \left[\Gamma^0_{00} + 2\Gamma^0_{0j}\frac{dx^j}{dt} + \Gamma^0_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\right]\frac{dx^i}{dt}. \tag{7}$$

If giving initial conditions, by numerical integrating the above equation, we can calculate out test particles' state at every time. Because of the symmetry of the space-time, number of nonzero components of Christoffel symbol $\Gamma^\mu_{\alpha\beta}$ is only fourteen. In addition, for the reason of existence of the integration constants, (7) just is a four-dimensional dynamical system in fact. Thus, the calculation is simplified greatly.

3 The Basic Theory of Gravitational Waves

First, we introduce the basic GWs theory briefly [22–24]. The most natural starting point for any discussion of GWs is linearized gravity. The space-time metric $g_{\mu\nu}$ can be treated as perturbation from a flat metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \|h_{\mu\nu}\| \ll 1, \tag{8}$$

where $\eta_{\mu\nu}$ is the famous Minkowski metric. Only to one order approximation, we find the Einstein tensor can be written as

$$G_{\mu\nu} = -\frac{1}{2}(\partial_\alpha\partial^\alpha\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} - \partial_\nu\partial^\alpha\bar{h}_{\mu\alpha} - \partial_\mu\partial^\alpha\bar{h}_{\alpha\nu}), \tag{9}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, named the trace-reversed perturbation (because $\bar{h}^{\mu}_{\mu} = -h$, where $h = \eta^{\alpha\beta}h_{\alpha\beta}$). If we use the Lorentz gauge that is commonly used in studies of radiation, namely

$$\partial^{\nu}\bar{h}_{\mu\nu} = 0. \tag{10}$$

Then, the Einstein tensor becomes

$$G_{\mu\nu} = -\frac{1}{2}\partial_{\alpha}\partial^{\alpha}\bar{h}_{\mu\nu} = -\frac{1}{2}\Box^2\bar{h}_{\mu\nu}, \tag{11}$$

where $\Box^2 = \nabla^2 - \partial_t^2$. So, the linearized Einstein field equation is therefore

$$\Box^2\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}, \tag{12}$$

in vacuum, this reduces to

$$\Box^2\bar{h}_{\mu\nu} = 0. \tag{13}$$

Equations (12, 13) just are wave equations. Furthermore, introducing TT (transverse traceless) gauge, we get $\bar{h}_{\mu\nu} = h_{\mu\nu}$, where $h_{\mu\nu}$ only has two independent polarization components: $h_{\times}, h_{+}$.

We can gain the solution of (12) by some calculation,

$$\bar{h}_{\mu\nu} = \frac{2}{r}\frac{d^2 Q_{\mu\nu}(t-R)}{dt^2}, \tag{14}$$

this is the quadrupole formula of gravitational radiation, where R represents the distance of waves source with observer. The retarded time $t - R$ shows that the propagation speed of GWs is the light-speed. And $Q_{\mu\nu}$ is the mass-energy distribution,

$$Q_{\mu\nu} = \int \rho x^{\alpha} x^{\beta} d^3x, \tag{15}$$

here $\rho = T^{tt}$ is the mass-energy density of source, x is the coordinate of source point.

The above mentioned can also be extended to a curve space-time. In our case, we can get [22]

$$Q_{ij} = \left(x_i x_j - \frac{1}{3}\delta_{ij}r^2\right), \quad x_i = (x, y, z). \tag{16}$$

The gravitational wave has two modes h_{+} and $h_{\times}$ which are given as [19]

$$\begin{aligned} h_{+} = \Bigg[&(h^Q_{xx} - h^Q_{yy})\frac{\cos^2\Theta + 1}{4}\cos 2\Phi - \frac{h^Q_{xx} + h^Q_{yy} - 2h^Q zz}{4}\sin^2\Theta \\ &+ h^Q_{xy}\frac{\cos^2\Theta + 1}{2}\sin 2\Phi - h^Q_{xz}\frac{\sin 2\Theta}{2}\cos\Phi - h^Q_{yz}\frac{\sin 2\Theta}{2}\sin\Phi\Bigg], \end{aligned} \tag{17}$$

and

$$h_{\times} = -\frac{h^Q_{xx} - h^Q_{yy}}{2}\cos\Theta\sin 2\Phi + h^Q_{xy}\cos\Theta\cos 2\Phi + h^Q_{xz}\sin\Theta\sin\Phi - h^Q_{yz}\sin\Theta\cos\Phi, \tag{18}$$

where Θ, Φ denote the observation point's latitude and azimuth respectively. And h^Q_{ij} is defined as

$$h^Q_{ij} = \frac{2}{R}\frac{d^2 Q_{ij}}{dt^2}, \tag{19}$$

where R stands for the distance between the observer and the source.

4 Gravitational Waveform from Free Test Particles

We can determine a set of valid initial conditions by parameterizing orbits using the energy and angular momentum, but it is more nature to think in terms of the orbital geometry: the pericenter $r_{\min}$, the apocenter $r_{\max}$ (can determine eccentricity), and the orbital inclination angle [28]. So, we research the emission of GWs from two different systems under same orbital geometry instead of energy and momentum.

Since the emission of gravitational radiation tends to circularize the orbits of the binary system [25], we study the GWs form under the assumption of circular orbits firstly. The conditions of circular orbits request: $\theta = \pi/2, \dot{r} = \dot{\theta} = 0, \ddot{r} = \ddot{\theta} = 0$ (here the dot represents derivation of τ). So, from (4–6), we get one constraint condition that

$$\frac{E^2}{g_{tt}} + \frac{L^2}{g_{\phi\phi}} = -1. \tag{20}$$

And $\ddot{r} = 0$ gives another condition

$$\frac{\partial g_{tt}}{\partial r}\frac{E^2}{g_{tt}} + \frac{\partial g_{\phi\phi}}{\partial r}\frac{L^2}{g_{\phi\phi}} = 0, \tag{21}$$

but $\ddot{\theta} = 0$ can't give new constraint.

So, once we determine the initial radii r, the motion of particles would be decided doubtless by the above two equations, and the orbits are stable (here we don't discuss the stability of circular orbits). The solutions of (20), (21) are

$$E = \sqrt{\left(1 - \frac{2}{r}\right)(1 + B^2 r^2)^2 \left(1 + \frac{B^2 r^2 (3 - 2r) - 1}{3 - r - 5B^2 r^2 + 3B^2 r^2}\right)}, \tag{22}$$

and

$$L = \pm \frac{r}{1 + B^2 r^2}\sqrt{\frac{B^2 r^2 (3 - 2r) - 1}{3 - r - 5B^2 r^2 + 3B^2 r^2}}. \tag{23}$$

Obviously, if $B = 0$, we would go back too the case of the Schwarzschild black hole.

The observer being assumed to be on the equatorial plane and $\Phi = 0$ (in this paper, all observer are in the same position). For a circular orbit, $r = \text{const}$, $\theta = \pi/2$ and $\dot{\phi} = \text{const}$. Then we get the waveform from (16, 17, 19)

$$R \cdot h_+ = -2r^2\omega^2 \cos 2\omega t, \tag{24}$$

where $\omega = Lg_{tt}/Eg_{\phi\phi}$. Note that $h_\times$ vanishes because the particle moves on the equatorial plane and the observer is also on the equatorial plane. We can find that ω with $B \neq 0$ is larger than the case of $B = 0$ (Schwarzschild) under same orbit radii r, so that we think

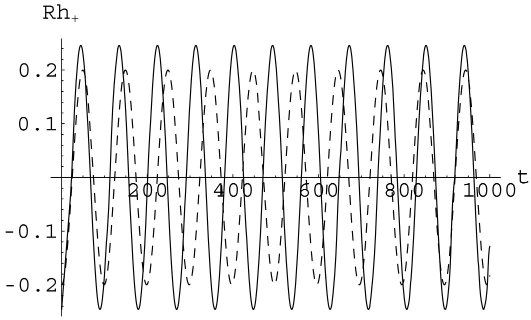

Fig. 1 The h_+ form of circle orbits. The *solid line* presents waveform of $B = 0.01$ and *dashed line* is the case of the Schwarzschild black hole. The radii of orbits both are 10. We can clearly find that amplitude and frequency become a little larger when existing magnitude field

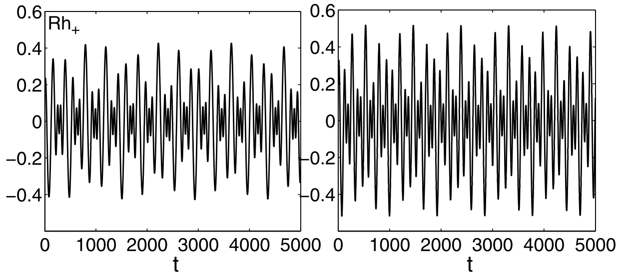

Fig. 2 The gravitational waveforms h_+ mode of test particles on the equatorial plane around the Schwarzschild black hole (*left*) and the black hole immersed in magnetic field with $B = 0.01$ (*right*). Though two GW forms are similar, we can find the amplitude becomes lager for the reason of magnetic field. Both the orbits have same parameters: $r_{\max} = 15$ and $r_{\min} = 6$

the uniform exterior magnetic field can enlarge amplitude and frequency of GWs for circle orbits. For an exhibition, the h_+ waveforms of $B = 0.01$ and $B = 0$ while radii of orbits $r = 10$ are plotted in Fig. 1.

Now, we consider the general orbits on the equatorial plane (the inclination is zero always). It is the same with the case of circle orbits, the $h_\times$ modes also equal 0. For the convenience of comparing two kind of black holes, we choose same orbit parameters: the maximal radii $r_{\max}$ and the minimum radii $r_{\min}$. The GWs forms which are pictured in Fig. 2 are very similar. But it looks likely that the GWs form has more large amplitude in the case of magnetic one. For the sake of more clear description, we plot the energy spectrum of GWs in Fig. 3. The energy spectrum of gravitational wave is one of the most important observable in gravitational astronomy. The energy spectrum of the GW is calculated as [19]

$$\frac{dE}{d\omega} \propto (|\hat{A}_+|^2 + |\hat{A}_\times|^2), \tag{25}$$

where $\hat{A}$ denotes the Fourier coefficients of each wave mode. In Fig. 5 the energy spectra for the cases of the pure Schwarzschild and magnetized one are shown. These are normalized by the peak value in the case of magnetized black hole. And we find that the magnetized one has more strong energy radiation, and shifts to higher frequency. Then we repeat the same process by varying pericenter $r_{\min}$ and eccentricity, find the results are changeless.

Lastly, we study the most ordinary orbits no restricted on the equatorial plane. Because the space-time immersed in magnetic field is no longer symmetric about equatorial plane, we are difficult to choose the same inclination with Schwarzschild one. We only give a rough comparison between these two cases. For the purpose of the test particle "breaking" out the equatorial plane, we use the same initial conditions in Figs. 2 and 3, but change tinily the value of $\dot{\theta}_0 = 0.005$. Thus the orbits of the test particles around the pure Schwarzschild black

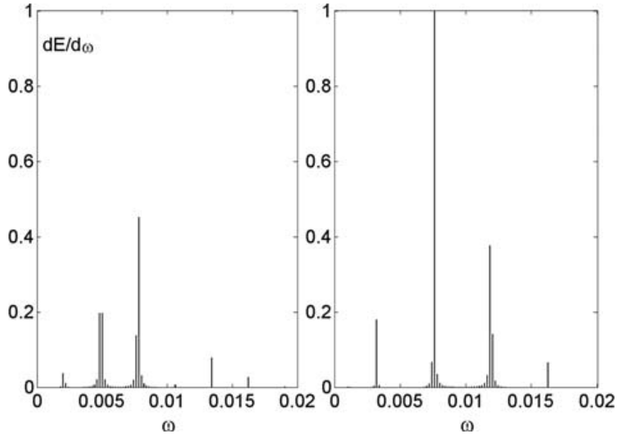

Fig. 3 Energy spectrum of GWs depicted in Fig. 2. The *left one* is the case of pure Schwarzschild, and the *right* is the case of the black hole immersed in magnetic field. It is clear that the magnetized one has more strong energy radiation, and shifts to higher frequency. Both the orbits have same parameters: $r_{\max} = 15$ and $r_{\min} = 6$

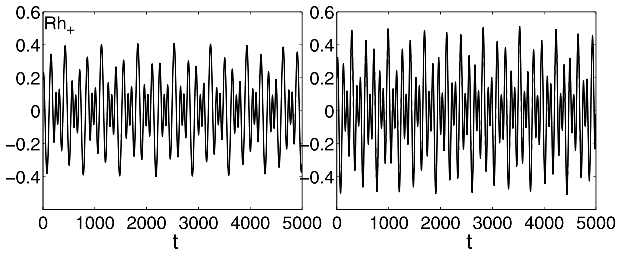

Fig. 4 The gravitational waveforms h_+ mode of test particles around (no longer on the equatorial palne) the Schwarzschild black hole (*left*) and the black hole immersed in magnetic field with $B = 0.01$ (*right*). Though two GW forms are similar, we can find the amplitude becomes lager while existing magnetic field. The initial conditions are as the same as Fig. 2 but $\dot{\theta}_0 = 0.005$

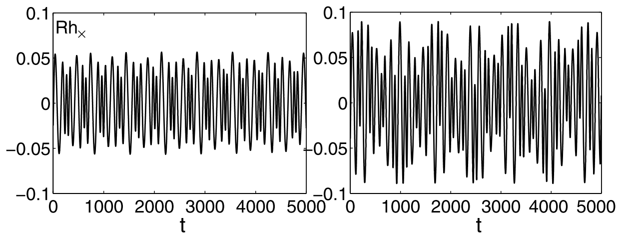

Fig. 5 The gravitational waveforms $h_\times$ mode of test particles around (no longer on the equatorial palne) the Schwarzschild black hole (*left*) and the black hole immersed in magnetic field with $B = 0.01$ (*right*). Not only the amplitude becomes lager while existing magnetic field, but two waveforms are no longer similar. The initial conditions are as the same as Fig. 4

hole and the black hole immersed in uniform magnetic field are very similar. So that we can roughly compare their GW forms. In Fig. 4, we plot the h_+ modes of two orbits, and we can get the same conclusion in Fig. 2.

In Fig. 5, we give the wave form of $h_\times$ mode, and find not only the amplitude became more large while existing magnetic field, but the shape of two waves are also obvious different. Furthermore, the energy spectrum of GWs radiation of two orbits are given in Fig. 6

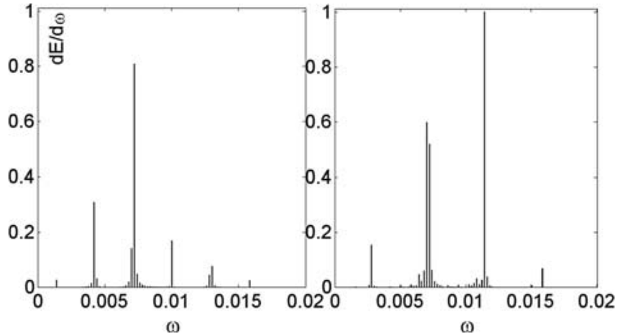

Fig. 6 Energy spectrum of GWs radiated from the most ordinary orbits no restricted on the equatorial plane. The *left one* is the case of pure Schwarzschild black hole, and the *right* is the case of the black hole immersed in the magnetic field. It is clearl that the magnetized one has more strong energy radiation, and shifts to higher frequency. The initial conditions are as the same as Fig. 4

too. The same conclusion can be obtained that the case of magnetic field has more strong energy radiation, and the energy radiation shifts to higher frequency. We vary the orbital parameters, and find the same results.

5 Discussion and Conclusions

For the sake of researching the role of magnetic field in gravitational waves radiation, we consider a simple model with a Schwarzschild black hole immersed in a uniform strong magnetic field. Because of existence of magnetic field, strength of gravitational field should be stronger than that in the Schwarzschild field. If orbits shape are the same, the amplitude of GWs from test particle around Ernst metric should be larger. But obtaining quantitative detail must need the help of numerical calculation. So we take three kind typical orbits: circle orbit, orbit restricted on the equatorial plane and general orbit. By comparison with the case of the pure Schwarzschild black hole with same orbit parameters, we find not only the amplitude of GWs from the case of magnetic one is bigger, but the energy spectrum also shifts to higher frequency. From the analytic expression of the gravitational waveform of circle orbit, this conclusion is very obvious. On the other hand, the uniform strong magnetic field can change the shape of $h_\times$ mode but not too obvious for h_+ one.

For firming our conclusion, we take many other orbits, and the conclusion is similar. In future, we plan to study the case of magnetized Kerr black holes.

Acknowledgements The author thanks very much to professor Xinhao Liao for his useful help.

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